

Algebraic model of an oblate top

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Abstract

We consider an algebraic treatment of a three-body system. In particular, we develop the formalism for a system of three identical objects and discuss an application to nonstrange baryon resonances which are interpreted as vibrational and rotational excitations of an oblate symmetric top. We derive closed expressions for a set of elementary form factors that appear in the calculation of both electromagnetic, strong and weak couplings of baryons.

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1 Introduction

The development of spectrum generating algebras and the study of exactly solvable systems have played an important role in all fields of physics [1]. In particular, in spectroscopic studies algebraic methods are very useful to study the symmetries and selection rules, to classify the basis states, and to calculate matrix elements. An exactly solvable model that is of special interest is the symmetric top. As an example, we mention a study by Bohm and Teese [2] in which the rotational spectrum of a prolate symmetric top was treated in terms of $SO(3, 2)$ with representation doubling by parity.

The aim of this contribution is to study a realization of a symmetric top in which all vibrational and rotational degrees of freedom are present from the outset. A good candidate for such an approach is provided by the so-called algebraic method that was developed for the collective vibrations and rotations of nuclei [3], and later extended to rovibrational states of molecules [4, 5, 6] and collective excitations of the nucleon [7]. This approach is based on the general criterion [8] to take $U(k + 1)$ as a spectrum generating algebra for a bound-state problem with k degrees of freedom and assigning all states to the symmetric representation $[N]$ of $U(k + 1)$. For collective nuclei this led to the introduction of the $U(6)$ interacting boson model [3] and for diatomic molecules to the $U(4)$ vibron model [4].

In this contribution we apply the algebraic method to a three-body system. The dynamics of this system is determined by the six degrees of freedom of the two relative coordinates, which in the algebraic approach leads to a $U(7)$ spectrum generating algebra. We develop the formalism for a system of three identical objects and discuss an application to nonstrange baryon resonances which are interpreted as vibrations and rotations of an oblate symmetric top.

2 Algebraic treatment of a three-body system

The geometry of a three-body system is characterized by two relative coordinates which we choose as the relative Jacobi coordinates

$$\begin{aligned}\vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2) , \\ \vec{\lambda} &= \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) .\end{aligned}\tag{2.1}$$

Here \vec{r}_1 , \vec{r}_2 and \vec{r}_3 are the coordinates of the three objects. Instead of using coordinates and momenta we prefer to use a second quantized formalism, in which we introduce a dipole boson for each independent relative coordinate and an auxiliary scalar boson

$$b_{\rho,m}^\dagger, b_{\lambda,m}^\dagger, s^\dagger \quad (m = -1, 0, 1) .\tag{2.2}$$

The scalar boson does not represent an independent degree of freedom, but is added under the restriction that the total number of bosons $N = n_\rho + n_\lambda + n_s$ is conserved. This procedure leads to a compact

spectrum generating algebra of $U(7)$ whose 49 generators are

$$b_{i,m}^\dagger b_{j,m'} , \quad b_{i,m}^\dagger s , \quad s^\dagger b_{i,m} , \quad s^\dagger s , \quad (2.3)$$

with $(m, m' = -1, 0, 1)$ and $(i, j = \rho, \lambda)$. The introduction of the scalar boson is just an elegant and efficient way by means of which the full dynamics of two vectors can be investigated. This includes situations in which there is a strong mixing of the oscillator basis (collective models). For a system of interacting bosons the model space is spanned by the symmetric irreps $[N]$ of $U(7)$, which contains the oscillator shells with $n = n_\rho + n_\lambda = 0, 1, 2, \dots, N$. The value of N determines the size of the model space.

For three identical objects (*e.g.* for X_3 molecules or nonstrange qqq baryons) the Hamiltonian (or mass operator) has to be invariant under the permutation group S_3 . The permutation symmetry of three identical objects is determined by the transposition $P(12)$ and the cyclic permutation $P(123)$ [9]. All other permutations can be expressed in terms of these two elementary ones. Algebraically, these operators can be expressed in terms of the generators $G_{ij} = \sum_m b_{i,m}^\dagger b_{j,m}$ that act in index space $(i, j = \rho, \lambda)$. We find

$$\begin{aligned} P(12) &= \exp \left\{ -i \frac{\pi}{2} (\hat{F}_3 + \hat{n}) \right\} = \exp \{ -i \pi \hat{n}_\rho \} , \\ P(123) &= \exp \left\{ -i \frac{2\pi}{3} \hat{F}_2 \right\} . \end{aligned} \quad (2.4)$$

The operators \hat{F}_2 and \hat{F}_3 are two components of the $SU(2)$ pseudo-spin in index space, which in turn is related to the vortex spin [10]

$$\begin{aligned} \hat{F}_1 &= \sum_m \left(b_{\rho,m}^\dagger b_{\lambda,m} + b_{\lambda,m}^\dagger b_{\rho,m} \right) , \\ \hat{F}_2 &= -i \sum_m \left(b_{\rho,m}^\dagger b_{\lambda,m} - b_{\lambda,m}^\dagger b_{\rho,m} \right) , \\ \hat{F}_3 &= \sum_m \left(b_{\rho,m}^\dagger b_{\rho,m} - b_{\lambda,m}^\dagger b_{\lambda,m} \right) . \end{aligned} \quad (2.5)$$

The operator \hat{n} counts the total number of dipole bosons

$$\hat{n} = \hat{n}_\rho + \hat{n}_\lambda = \sum_m \left(b_{\rho,m}^\dagger b_{\rho,m} + b_{\lambda,m}^\dagger b_{\lambda,m} \right) . \quad (2.6)$$

Since the permutation group S_3 is isomorphic to D_3 , one can either use the irreducible representations of S_3 or those of D_3 to label the three symmetry classes. In this contribution we use the D_3 labeling: A_1 and A_2 for the one-dimensional symmetric and antisymmetric representations, and E for the two-dimensional representation.

All operators of interest are expressed in terms of the building blocks of Eq. (2.2) which transform under D_3 as E_ρ , E_λ and A_1 , respectively. With the help of Eq. (2.4) one can construct physical operators with the appropriate symmetry properties. The most general form of the Hamiltonian, that preserves

angular momentum and parity, transforms as a scalar (*i.e.* A_1) under D_3 and contains at most two-body interactions is given by

$$\begin{aligned}
\hat{H} = & H_0 + \epsilon_s s^\dagger \tilde{s} - \epsilon_p (b_\rho^\dagger \cdot \tilde{b}_\rho + b_\lambda^\dagger \cdot \tilde{b}_\lambda) + u_0 (s^\dagger s^\dagger \tilde{s} \tilde{s}) - u_1 s^\dagger (b_\rho^\dagger \cdot \tilde{b}_\rho + b_\lambda^\dagger \cdot \tilde{b}_\lambda) \tilde{s} \\
& + v_0 \left[(b_\rho^\dagger \cdot b_\rho^\dagger + b_\lambda^\dagger \cdot b_\lambda^\dagger) \tilde{s} \tilde{s} + s^\dagger s^\dagger (\tilde{b}_\rho \cdot \tilde{b}_\rho + \tilde{b}_\lambda \cdot \tilde{b}_\lambda) \right] + c_1 (b_\rho^\dagger \times b_\lambda^\dagger)^{(1)} \cdot (\tilde{b}_\lambda \times \tilde{b}_\rho)^{(1)} \\
& + \sum_{l=0,2} c_l \left[(b_\rho^\dagger \times b_\rho^\dagger - b_\lambda^\dagger \times b_\lambda^\dagger)^{(l)} \cdot (\tilde{b}_\rho \times \tilde{b}_\rho - \tilde{b}_\lambda \times \tilde{b}_\lambda)^{(l)} + 4 (b_\rho^\dagger \times b_\lambda^\dagger)^{(l)} \cdot (\tilde{b}_\lambda \times \tilde{b}_\rho)^{(l)} \right] \\
& + \sum_{l=0,2} w_l (b_\rho^\dagger \times b_\rho^\dagger + b_\lambda^\dagger \times b_\lambda^\dagger)^{(l)} \cdot (\tilde{b}_\rho \times \tilde{b}_\rho + \tilde{b}_\lambda \times \tilde{b}_\lambda)^{(l)} .
\end{aligned} \tag{2.7}$$

Here $\tilde{b}_{\rho,m} = (-1)^{1-m} b_{\rho,-m}$, $\tilde{b}_{\lambda,m} = (-1)^{1-m} b_{\lambda,-m}$ and $\tilde{s} = s$. The dots indicate scalar products and the crosses tensor products. The Hamiltonian of Eq. (2.7) contains several interesting limiting situations. The harmonic oscillator model arises for $v_0 = 0$, *i.e.* no coupling between different harmonic oscillator shells. This scenario corresponds to the $U(7) \supset U(6)$ group reduction. On the other hand, for the choice $v_0 \neq 0$ the eigenfunctions are collective in nature, since they are spread over many oscillator shells.

The eigenvalues and corresponding eigenvectors can be obtained exactly by diagonalization in an appropriate basis. The wave functions obtained in this way have by construction good angular momentum L , parity P , and permutation symmetry t .

3 Permutation symmetry

The permutation symmetry of a given wave function can be determined from the transformation properties under $P(12)$ and $P(123)$ [9]. Since the Hamiltonian of Eq. (2.7) is invariant under the transposition $P(12)$, basis states with n_ρ even and n_ρ odd do not mix, and can therefore be treated separately. This allows one to distinguish wave functions with $t = A_1$ or E_λ from wave functions with $t = A_2$ or E_ρ

$$P(12) \begin{pmatrix} |\psi_{A_1}\rangle \\ |\psi_{A_2}\rangle \\ |\psi_{E_\rho}\rangle \\ |\psi_{E_\lambda}\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_{A_1}\rangle \\ |\psi_{A_2}\rangle \\ |\psi_{E_\rho}\rangle \\ |\psi_{E_\lambda}\rangle \end{pmatrix} . \tag{3.1}$$

The cyclic permutation $P(123)$ can be used to distinguish wave functions with $t = A_1$ or A_2 from wave functions with $t = E_\lambda$ or E_ρ

$$P(123) \begin{pmatrix} |\psi_{A_1}\rangle \\ |\psi_{A_2}\rangle \\ |\psi_{E_\rho}\rangle \\ |\psi_{E_\lambda}\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(2\pi/3) & -\sin(2\pi/3) \\ 0 & 0 & \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \begin{pmatrix} |\psi_{A_1}\rangle \\ |\psi_{A_2}\rangle \\ |\psi_{E_\rho}\rangle \\ |\psi_{E_\lambda}\rangle \end{pmatrix} . \tag{3.2}$$

In Eqs. (3.1) and (3.2) we have used that the application of $P(12)$ and $P(123)$ on $|\psi_t\rangle$ means carrying out the inverse operation on the boson operators, $b_{\rho,m}^\dagger$ and $b_{\lambda,m}^\dagger$ [9, 11]. In practice, the wave functions $|\psi_{E_\rho}\rangle$ and $|\psi_{E_\lambda}\rangle$ are obtained from separate diagonalizations, and hence are determined up to a sign. Eq. (3.2) can be used to determine their relative sign, so that they transform as the two components of the mixed symmetry doublet with $t = E$ symmetry.

In addition to angular momentum, parity and permutation symmetry, L_t^P , the Hamiltonian of Eq. (2.7) has another symmetry. Since \hat{H} commutes with the operator \hat{F}_2 of Eq. (2.5), its eigenstates can also be labeled by the eigenvalues of \hat{F}_2 : $m_F = 0, \pm 1, \dots, \pm N$. We denote these eigenstates by $|\phi_{m_F}\rangle$. Since the transposition $P(12)$ anticommutes with \hat{F}_2

$$P(12) \hat{F}_2 = -\hat{F}_2 P(12), \quad (3.3)$$

the simultaneous eigenfunctions of $P(12)$ and \hat{H} are given by the linear combinations

$$\begin{aligned} |\psi_1\rangle &= \frac{-i}{\sqrt{2(1+\delta_{m_F,0})}} [|\phi_{+m_F}\rangle - |\phi_{-m_F}\rangle], \\ |\psi_2\rangle &= \frac{(-1)^\nu}{\sqrt{2(1+\delta_{m_F,0})}} [|\phi_{+m_F}\rangle + |\phi_{-m_F}\rangle]. \end{aligned} \quad (3.4)$$

Here we have introduced the label ν by $m_F = \nu \pmod{3}$. These wave functions satisfy

$$P(12) \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}. \quad (3.5)$$

The cyclic permutation $P(123)$ acts in Fock space as a rotation generated by \hat{F}_2 (see Eq. (2.4))

$$P(123) \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \begin{pmatrix} \cos(2\pi m_F/3) & (-1)^\nu \sin(2\pi m_F/3) \\ -(-1)^\nu \sin(2\pi m_F/3) & \cos(2\pi m_F/3) \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}. \quad (3.6)$$

According to Eq. (3.2) the wave functions $|\psi_1\rangle$ ($|\psi_2\rangle$) transform for $\nu = 0$ as $t = A_2$ (A_1), and for $\nu = 1, 2$ as $t = E_\rho$ (E_λ). The special connection between the label $m_F = \nu \pmod{3}$ and the permutation symmetry only holds for Hamiltonians that commute with \hat{F}_2 . All one- and two-body D_3 invariant interactions indeed satisfy this property, and hence their eigenstates can be labeled by $M_F = |m_F|$ and L_t^P . However, at the three-body level there are D_3 invariant interactions that mix states with $\Delta m_F = \pm 6$ [6, 12]. The permutation symmetry can then still be determined by the general procedure outlined in Eqs. (3.1) and (3.2).

4 Oblate symmetric top

An analysis of the equilibrium shape of the potential energy surface that corresponds to Eq. (2.7) yields that the only stable nonlinear configuration is that of an equilateral triangle [13]. This equilibrium

configuration is represented in $U(7)$ by an intrinsic (or coherent) state in the form of a condensate

$$\frac{1}{\sqrt{N!}}(b_c^\dagger)^N |0\rangle, \quad (4.1)$$

with

$$b_c^\dagger = \left[s^\dagger + R(b_{\rho,y}^\dagger + b_{\lambda,x}^\dagger)/\sqrt{2} \right] / \sqrt{1 + R^2}. \quad (4.2)$$

The equilibrium shape of an equilateral triangle is a result of the underlying D_3 symmetry.

In order to analyze the vibrational and rotational excitations it is convenient to split the Hamiltonian into an intrinsic (vibrational) and a collective (rotational) part [14]. The intrinsic part of Eq. (2.7) by definition annihilates the condensate of Eqs. (4.1) and (4.2) and is given by [7]

$$\begin{aligned} \hat{H}_{\text{int}} = & \xi_1 \left(R^2 s^\dagger s^\dagger - b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger \right) \left(R^2 \tilde{s}\tilde{s} - \tilde{b}_\rho \cdot \tilde{b}_\rho - \tilde{b}_\lambda \cdot \tilde{b}_\lambda \right) \\ & + \xi_2 \left[\left(b_\rho^\dagger \cdot b_\rho^\dagger - b_\lambda^\dagger \cdot b_\lambda^\dagger \right) \left(\tilde{b}_\rho \cdot \tilde{b}_\rho - \tilde{b}_\lambda \cdot \tilde{b}_\lambda \right) + 4 \left(b_\rho^\dagger \cdot b_\lambda^\dagger \right) \left(\tilde{b}_\lambda \cdot \tilde{b}_\rho \right) \right]. \end{aligned} \quad (4.3)$$

For the special case of $R^2 = 0$, the intrinsic Hamiltonian has $U(7) \supset U(6)$ symmetry and corresponds to an anharmonic vibrator, whereas for $R^2 = 1$ and $\xi_2 = 0$ it has $U(7) \supset SO(7)$ symmetry and corresponds to a deformed oscillator.

In the more general case with $R^2 \neq 0$ and $\xi_1, \xi_2 > 0$, the intrinsic Hamiltonian of Eq. (4.3) describes the vibrational excitations of an oblate symmetric top. This can be seen from a normal mode analysis. To leading order in N , \hat{H}_{int} reduces to a harmonic form [6, 7]

$$\hat{H}_{\text{int}} = \epsilon_1 b_u^\dagger b_u + \epsilon_2 (b_v^\dagger b_v + b_w^\dagger b_w), \quad (4.4)$$

with eigenfrequencies $\epsilon_1 = 4N\xi_1 R^2$ and $\epsilon_2 = 4N\xi_2 R^2/(1 + R^2)$. The deformed bosons are given by

$$\begin{aligned} b_u^\dagger &= \left[-R s^\dagger + (b_{\rho,y}^\dagger + b_{\lambda,x}^\dagger)/\sqrt{2} \right] / \sqrt{1 + R^2}, \\ b_v^\dagger &= (b_{\lambda,x}^\dagger - b_{\rho,y}^\dagger)/\sqrt{2}, \\ b_w^\dagger &= (b_{\lambda,y}^\dagger + b_{\rho,x}^\dagger)/\sqrt{2}. \end{aligned} \quad (4.5)$$

The first term in Eq. (4.4) represents the symmetric stretching mode (b_u) and the second term a degenerate doublet of an antisymmetric stretching mode (b_v) and a bending mode (b_w). This is in agreement with the point-group classification of the fundamental vibrations of a symmetric X_3 configuration [15]. Therefore, the deformed bosons of Eqs. (4.2) and (4.5) correspond to a geometry of an oblate symmetric top with the threefold symmetry axis along the z -axis. In the large N limit the vibrational spectrum is harmonic

$$E_{\text{vib}} = \epsilon_1 v_1 + \epsilon_2 v_2. \quad (4.6)$$

In a geometric description, the excitations of an oblate top are labeled by $(v_1, v_2^l); K, L_t^P, M$. Here v_1 denotes the number of quanta in the symmetric stretching mode which has A_1 symmetry, and v_2 the total

number of quanta in the asymmetric stretching and the bending modes, which form a degenerate doublet with E symmetry. The label l is associated with the degenerate vibration. It is proportional to the vibrational angular momentum about the axis of symmetry and can have the values $l = v_2, v_2 - 2, \dots, 1$ or 0 for v_2 odd or even, respectively. The rotational states, which are characterized by the angular momentum L and its projection K on the three-fold symmetry axis, are arranged in bands built on top of each vibration. The projection K can take the values $K = 0, 1, 2, \dots$, while the values of the angular momentum are $L = K, K + 1, K + 2, \dots$. The parity is $P = (-)^K$, t denotes the transformation character of the total wave function under D_3 , and M is the angular momentum projection. For a given value of l and K the degeneracy of a state with angular momentum L is given by $4(2L + 1)/(1 + \delta_{l,0})(1 + \delta_{K,0})$.

5 Geometric interpretation of M_F

In section 3 we showed that the eigenstates of the algebraic Hamiltonian of Eq. (2.7) can be labeled by M_F, L_t^P . The same holds for its intrinsic part, Eq. (4.3), which describes the vibrational excitations of an oblate top. In this section we wish to elucidate the role of the label M_F in the context of the oblate symmetric top, and, in particular, its relation to the geometric labels K and l .

The connection between an algebraic and a geometric description of an oblate top can be studied by means of intrinsic (or coherent) states [16]. In such an approach, each vibrational band (v_1, v_2^l) is represented by an intrinsic state which can be obtained from Eq. (4.1) by replacing a condensate boson (b_c^\dagger) by one of the deformed bosons of Eq. (4.5). However, the deformed operators, b_v^\dagger and b_w^\dagger , do not have good projection of the vibrational angular momentum on the symmetry axis. In order to construct intrinsic states with well-defined projection on the symmetry axis, we transform the cartesian bosons of Eq. (4.5) to spherical bosons, and introduce the linear combinations [9]

$$\begin{aligned}\eta_m^\dagger &= (b_{\lambda,m}^\dagger + i b_{\rho,m}^\dagger)/\sqrt{2}, \\ \zeta_m^\dagger &= (b_{\lambda,m}^\dagger - i b_{\rho,m}^\dagger)/\sqrt{2},\end{aligned}\tag{5.1}$$

for which we have

$$\begin{aligned}\eta_1^\dagger &= (-b_v^\dagger - i b_w^\dagger)/\sqrt{2}, \\ \zeta_{-1}^\dagger &= (b_v^\dagger - i b_w^\dagger)/\sqrt{2}.\end{aligned}\tag{5.2}$$

In this representation, the operator \hat{F}_2 has the simple form of the difference between two number operators

$$\hat{F}_2 = \hat{n}_\zeta - \hat{n}_\eta.\tag{5.3}$$

The intrinsic state for a vibration (v_1, v_2^l) with projection $\pm l$ on the symmetry axis is then given by (for

$N \rightarrow \infty$)

$$|N, v_1, v_2, \pm l; R\rangle = \frac{(\eta_1^\dagger)^{(v_2 \pm l)/2}}{\sqrt{((v_2 \pm l)/2)!}} \frac{(\zeta_{-1}^\dagger)^{(v_2 \mp l)/2}}{\sqrt{((v_2 \mp l)/2)!}} \frac{(b_u^\dagger)^{v_1}}{\sqrt{v_1!}} \frac{(b_c^\dagger)^{N-v_1-v_2}}{\sqrt{(N-v_1-v_2)!}} |0\rangle, \quad (5.4)$$

which can be expressed in terms of a sum over (n_η, n_ζ) configurations

$$(\eta_1^\dagger)^{(v_2 \pm l)/2} (\zeta_{-1}^\dagger)^{(v_2 \mp l)/2} (\eta_{-1}^\dagger)^{n_\eta - (v_2 \pm l)/2} (\zeta_1^\dagger)^{n_\zeta - (v_2 \mp l)/2} (s^\dagger)^{N - n_\eta - n_\zeta} |0\rangle. \quad (5.5)$$

For each of these configurations the projection K of the angular momentum along the symmetry axis is

$$K = n_\zeta - n_\eta \pm 2l = m_F \pm 2l. \quad (5.6)$$

This shows that the algebraic label m_F has a direct interpretation in terms of the geometric labels, K and l . For $l = 0$ we have $m_F = K$, but for $l > 0$ there are two possible values of m_F for each K . Thus $M_F = |m_F| = |K \mp 2l|$ provides an additional quantum number which is needed for a complete classification of the rotational excitations of an oblate top. For example, the $(v_1, v_2^{l=1})$ vibrational band has two L_E^- levels which have the same value of $K = 3$, but different values of M_F . The rotational spectrum is given by

$$\begin{aligned} E_{\text{rot}} &= \kappa_1 L(L+1) - \kappa_2 M_F^2 \\ &= \kappa_1 L(L+1) - \kappa_2 (K^2 \mp 4Kl + 4l^2). \end{aligned} \quad (5.7)$$

The last term contains the effects of the Coriolis force which gives rise to a $8\kappa_2 Kl$ splitting of the rotational levels which increases linearly with K . The label M_F plays a role similar to that of the label G discussed by Watson [12] for X_3 molecules, and that of the label m by Bowler *et al.* [17] in the context of a harmonic oscillator quark model in baryon spectroscopy. As an example of the assignments of K and M_F we show in Figures 1 and 2 the classification scheme for the levels with $L \leq 3$ belonging to a (v_1, v_2^l) vibrational band with $l = 0$ and $l = 1$, respectively.

With the exception of the levels with $K = l = 0$, all levels in Figures 1 and 2 are doubly degenerate. According to Eq.(5.7) the splitting of the levels with $K = 0$ is zero, whereas for $K > 0$ there remains a double degeneracy because of the two projections $\pm K$ on the symmetry axis. In particular, the rotational spectrum of Figure 2 does not exhibit l -type doubling, and therefore for $M_F = 0 \pmod{3}$ there is a degenerate doublet consisting of A_1 and A_2 levels. This is a consequence of the fact that the one- and two-body D_3 invariant Hamiltonians of Eqs. (2.7) and (4.3) commute with \hat{F}_2 . The degeneracy of the A_1 and A_2 states can be lifted by introducing higher order interactions that break M_F symmetry. For example, there exist three-body D_3 invariant interactions that mix states with $\Delta M_F = \pm 6$ [6]. A similar situation is encountered in Watson's effective Hamiltonian [12], whose main terms are diagonal in the quantum number G (which plays a similar role as M_F), but which also contains small higher-order corrections with $\Delta G = \pm 6$.

We note that $U(7)$ can also accommodate other types of rotations (*e.g.* nonrigid) and kinematics (*e.g.* relativistic). In such cases, the $L(L+1)$ and M_F^2 terms in Eq. (5.7) will be replaced by a general function of the angular momentum L and M_F [7].

6 Wave functions

In the algebraic approach, the oblate top wave functions can be obtained by projection from an intrinsic state. In principle this is an exact procedure, but since the expressions for the intrinsic states of Eq. (5.4) are only valid for $N \rightarrow \infty$, the same holds for the results presented in this section. In the large N limit we find

$$|N, v_1, v_2, l; R; K, L, M\rangle = \sqrt{\frac{2L+1}{8\pi^2}} \int d\Omega \mathcal{D}_{MK}^{(L)*}(\Omega) |N, v_1, v_2, l; R, \Omega\rangle . \quad (6.1)$$

The angle Ω specifies the orientation of the intrinsic state

$$\begin{aligned} |N, v_1, v_2, l; R, \Omega\rangle &= \mathcal{R}(\Omega) |N, v_1, v_2, l; R\rangle , \\ \mathcal{R}(\Omega) &= e^{-i\psi \hat{L}_z} e^{-i\theta \hat{L}_y} e^{-i\phi \hat{L}_z} . \end{aligned} \quad (6.2)$$

Next we construct states with good D_3 symmetry by considering the the action of the transposition and the cyclic permutation on the projected wave function

$$\begin{aligned} P(12) |N, v_1, v_2, l; R; K, L, M\rangle &= (-1)^{v_2+L} |N, v_1, v_2, -l; R; -K, L, M\rangle , \\ P(123) |N, v_1, v_2, l; R; K, L, M\rangle &= e^{2\pi i(K-2l)/3} |N, v_1, v_2, l; R; K, L, M\rangle . \end{aligned} \quad (6.3)$$

Here we take without loss of generality $K \geq 0$ and $l \geq 0$. States with good D_3 symmetry are given by the linear combinations

$$\begin{aligned} |\psi_1\rangle &= \frac{-i}{\sqrt{2(1+\delta_{K,0}\delta_{l,0})}} \left[|N, v_1, v_2, -l; R; K, L, M\rangle - (-1)^{v_2+L} |N, v_1, v_2, l; R; -K, L, M\rangle \right] , \\ |\psi_2\rangle &= \frac{(-1)^\nu}{\sqrt{2(1+\delta_{K,0}\delta_{l,0})}} \left[|N, v_1, v_2, -l; R; K, L, M\rangle + (-1)^{v_2+L} |N, v_1, v_2, l; R; -K, L, M\rangle \right] . \end{aligned} \quad (6.4)$$

These wave functions are characterized by $m_F = K + 2l = \nu \pmod{3}$. For $K \neq 0$ and $l \neq 0$ there are two extra linear combinations with $m_F = K - 2l = \nu \pmod{3}$

$$\begin{aligned} |\psi_3\rangle &= \frac{-i}{\sqrt{2(1+\delta_{K,0}\delta_{l,0})}} \left[|N, v_1, v_2, l; R; K, L, M\rangle - (-1)^{v_2+L} |N, v_1, v_2, -l; R; -K, L, M\rangle \right] , \\ |\psi_4\rangle &= \frac{(-1)^\nu}{\sqrt{2(1+\delta_{K,0}\delta_{l,0})}} \left[|N, v_1, v_2, l; R; K, L, M\rangle + (-1)^{v_2+L} |N, v_1, v_2, -l; R; -K, L, M\rangle \right] . \end{aligned} \quad (6.5)$$

According to Eq. (3.2) the wave functions $|\psi_1\rangle$ ($|\psi_2\rangle$) and $|\psi_3\rangle$ ($|\psi_4\rangle$) transform for $\nu = 0$ as $t = A_2$ (A_1), and for $\nu = 1, 2$ as $t = E_\rho$ (E_λ). These wave functions are consistent with the choice of geometry in the deformed bosons of Eqs. (4.2) and (4.5).

7 Baryon resonances

In this section we discuss an application of the oblate top model in baryon spectroscopy. We consider baryons to be built of three constituent parts (quarks or otherwise) with the string configuration of Fig. 3. The full algebraic structure is obtained by combining the geometric part, $U(7)$, with the internal spin-flavor-color part, $SU_{sf}(6) \otimes SU_c(3)$ (not considering heavy quarks),

$$\mathcal{G} = U(7) \otimes SU_{sf}(6) \otimes SU_c(3) . \quad (7.1)$$

For the nucleon (isospin $I = 1/2$) and delta ($I = 3/2$) families of resonances the three strings of Fig. 3 have equal lengths and equal relative angles. The three constituent parts form an equilateral triangle with $D_{3h} \supset D_3$ point group symmetry. Baryon resonances are then interpreted in terms of vibrations and rotations of an oblate symmetric top. In order to have total baryon wave functions that are antisymmetric, the permutation symmetry of the geometric (or spatial) part must be the same as the permutation symmetry of the spin-flavor part (the color part is a color singlet, *i.e.* antisymmetric). Therefore one can also use the dimension of the $SU_{sf}(6)$ representations to label the states: $A_1 \leftrightarrow [56]$, $A_2 \leftrightarrow [20]$ and $E \leftrightarrow [70]$. The nucleon itself is identified with the oblate top ground state $(v_1, v_2^l); K, L_t^P = (0, 0^0); 0, 0_{A_1}^+$, whereas the $N(1440)$ Roper and the $N(1710)$ resonances are interpreted as one-phonon excitations with $(1, 0^0); 0, 0_{A_1}^+$ and $(0, 1^1); 0, 0_E^+$, respectively (see Figure 4).

In such a collective model of baryons, the mass operator is written in terms of a spatial and a spin-flavor part. The spatial part is given by Eq. (4.3) plus a term linear in the angular momentum L to reproduce the linear Regge trajectories, while the spin-flavor part is given by the Gürsey-Radicati form [18]. With this mass operator we obtained a good overall fit of the spectrum of the nucleon and delta resonances with a r.m.s. deviation of 39 MeV [7, 13].

A far more sensitive test of models of baryon structure is provided by electromagnetic, strong and weak couplings. In the next section we show how the general formalism of the oblate top, which was developed in the previous sections, can be used to derive closed expressions for helicity amplitudes that can be measured in photo- and electroproduction and strong decays of baryon resonances.

8 Form factors

Helicity amplitudes for electromagnetic and strong couplings can be expressed in terms of some elementary spatial matrix elements or form factors. For nonstrange resonances these are the matrix elements of the operators [7, 20, 21]

$$\begin{aligned}\hat{U} &= e^{-ik\beta\hat{D}_{\lambda,z}/X_D}, \\ \hat{T}_{\lambda,m} &= \frac{im_3k_0\beta}{2X_D} \left(\hat{D}_{\lambda,m} e^{-ik\beta\hat{D}_{\lambda,z}/X_D} + e^{-ik\beta\hat{D}_{\lambda,z}/X_D} \hat{D}_{\lambda,m} \right), \\ \hat{T}_{\rho,m} &= \frac{im_3k_0\beta}{2X_D} \left(\hat{D}_{\rho,m} e^{-ik\beta\hat{D}_{\lambda,z}/X_D} + e^{-ik\beta\hat{D}_{\lambda,z}/X_D} \hat{D}_{\rho,m} \right),\end{aligned}\quad (8.1)$$

with $(m = -1, 0, 1)$. Here $\vec{k} = k\hat{z}$ is the photon (meson) momentum, k_0 is the photon (meson) energy, m_3 is the constituent mass and β represents the scale of the coordinate. The operators $\hat{D}_{\rho,m}$ and $\hat{D}_{\lambda,m}$ are dipole operators in $U(7)$

$$\begin{aligned}\hat{D}_{\rho,m} &= (b_\rho^\dagger \times \tilde{s} - s^\dagger \times \tilde{b}_\rho)_m^{(1)}, \\ \hat{D}_{\lambda,m} &= (b_\lambda^\dagger \times \tilde{s} - s^\dagger \times \tilde{b}_\lambda)_m^{(1)}.\end{aligned}\quad (8.2)$$

The normalization factor X_D is given by the reduced matrix element between the $L_t^P = 0_{A_1}^+$ ground state and the first excited 1_E^- state (both belonging to the $(v_1, v_2) = (0, 0^0)$ ground band)

$$X_D = \langle 1_E^- || \hat{D}_\lambda || 0_{A_1}^+ \rangle = -\langle 1_E^- || \hat{D}_\rho || 0_{A_1}^+ \rangle. \quad (8.3)$$

In the large N limit (and $R^2 > 0$) it reduces to

$$\lim_{N \rightarrow \infty} X_D = \frac{NR\sqrt{2}}{1+R^2}. \quad (8.4)$$

Since \hat{D}_λ is a generator of the algebra of $U(7)$, the matrix elements of \hat{U} are representation matrix elements of $U(7)$, *i.e.* a generalization of the Wigner \mathcal{D} -matrices for $SU(2)$. By making an appropriate basis transformation they can be obtained numerically without having to make any further approximations. However, in the limit of $N \rightarrow \infty$ (infinitely large model space) the matrix elements of \hat{U} , $\hat{T}_{\lambda,m}$ and $\hat{T}_{\rho,m}$ can also be derived in closed form.

We illustrate the method by evaluating the matrix elements of \hat{U} that connect the nucleon with its excited rotational states. In the collective model, the nucleon wave function is that of the ground state of the oblate symmetric top

$$|\psi_0\rangle = |N, v_1 = 0, v_2 = 0, l = 0; R; K = 0, L = 0, M = 0\rangle. \quad (8.5)$$

The vibrationally elastic matrix element connecting the ground state with a rotational excitation is given by

$$\begin{aligned}&\langle N, 0, 0, 0; R; K, L, M | \hat{U} | N, 0, 0, 0; R; 0, 0, 0 \rangle \\ &= \frac{\sqrt{2L+1}}{8\pi^2} \int d\Omega d\Omega' \mathcal{D}_{MK}^{(L)}(\Omega) \mathcal{D}_{00}^{(0)*}(\Omega') \langle N, 0, 0, 0; R, \Omega | \hat{U} | N, 0, 0, 0; R, \Omega' \rangle.\end{aligned}\quad (8.6)$$

In the large N limit the matrix element appearing in the integrand becomes diagonal in the orientation Ω of the condensate. The remaining angular integral can be obtained in closed form in terms of a spherical Bessel function

$$\begin{aligned} & \frac{\sqrt{2L+1}}{8\pi^2} \int d\Omega \mathcal{D}_{MK}^{(L)}(\Omega) e^{i k\beta \sin \theta \cos \phi} \\ &= \delta_{M,0} i^K \sqrt{2L+1} \frac{\sqrt{(L+K)!(L-K)!}}{(L+K)!!(L-K)!!} \frac{1}{2} [1 + (-1)^{L-K}] j_L(k\beta) . \end{aligned} \quad (8.7)$$

With the wave functions of Eq. (6.4) we find that the matrix elements connecting the nucleon with its rotationally excited states are

$$\begin{aligned} \langle \psi_1 | \hat{U} | \psi_0 \rangle &= 0 , \\ \langle \psi_2 | \hat{U} | \psi_0 \rangle &= \delta_{M,0} (-1)^\nu i^K \sqrt{\frac{2L+1}{2(1+\delta_{K,0})}} \frac{\sqrt{(L+K)!(L-K)!}}{(L+K)!!(L-K)!!} [1 + (-1)^{L-K}] j_L(k\beta) . \end{aligned} \quad (8.8)$$

For the $(v_1, v_2^l) = (0, 0^0)$ ground band we have $K = \nu \pmod{3}$.

To summarize the results for the matrix elements of Eq. (8.1) we introduce the notation

$$\begin{aligned} F_i(k) &= \langle \psi_i | \hat{U} | \psi_0 \rangle , \\ G_{i;\lambda,m}(k) &= \langle \psi_i | \hat{T}_{\lambda,m} | \psi_0 \rangle , \\ G_{i;\rho,m}(k) &= \langle \psi_i | \hat{T}_{\rho,m} | \psi_0 \rangle , \end{aligned} \quad (8.9)$$

for the elementary oblate top form factors. The nucleon wave function $|\psi_0\rangle$ is given in Eq. (8.5), and the wave function of the resonance $|\psi_i\rangle$ is given in Eq. (6.4) for $i = 1, 2$ and in Eq. (6.5) for $i = 3, 4$. All matrix elements can be expressed in terms of spherical Bessel functions. For the rotational transitions we find

$$\begin{aligned} F_1(k) &= 0 , \\ F_2(k) &= \delta_{M,0} Z(k\beta) , \end{aligned} \quad (8.10)$$

where $Z(k\beta)$ can be obtained by comparing with Eq. (8.8). The matrix elements of $\hat{T}_{\lambda,m}$ and $\hat{T}_{\rho,m}$ can be derived as

$$\begin{aligned} G_{1;\lambda,m}(k) &= 0 , \\ G_{2;\lambda,z}(k) &= -\delta_{M,0} m_3 k_0 \beta \frac{dZ(k\beta)}{dk\beta} , \\ G_{2;\lambda,\pm}(k) &= \pm \delta_{M,\pm 1} m_3 k_0 \beta \sqrt{L(L+1)} \frac{Z(k\beta)}{k\beta} , \end{aligned} \quad (8.11)$$

and

$$G_{1;\rho,z}(k) = -\delta_{M,0} m_3 k_0 \beta K(-1)^\nu \frac{Z(k\beta)}{k\beta} ,$$

$$\begin{aligned}
G_{1;\rho,\pm}(k) &= \pm \delta_{M,\pm 1} m_3 k_0 \beta i^K \sqrt{\frac{2L+1}{2(1+\delta_{K,0})}} \frac{\sqrt{(L+K)!(L-K)!}}{(L+K)!!(L-K)!!} \frac{K}{\sqrt{L(L+1)}} \\
&\times [1 + (-1)^{L-K}] \frac{1}{2L+1} [(L+1) j_{L-1}(k\beta) - L j_{L+1}(k\beta)] \\
&+ \delta_{M,\pm 1} m_3 k_0 \beta i^K \sqrt{\frac{2L+1}{2(1+\delta_{K,0})}} \frac{\sqrt{(L+K)!(L-K)!}}{(L+K-1)!!(L-K-1)!!} \frac{1}{\sqrt{L(L+1)}} \\
&\times [1 - (-1)^{L-K}] j_L(k\beta) , \\
G_{2;\rho,m}(k) &= 0 .
\end{aligned} \tag{8.12}$$

In Eqs. (8.11) and (8.12) we have used $G_{\cdot,\cdot,\pm}(k) = \mp \sqrt{2} G_{\cdot,\cdot,\pm 1}(k)$. We see that all relevant matrix elements can be expressed in terms of $F_2(k)$ and $G_{1;\rho,\pm}(k)$ only. In Table I we present the results for the elementary form factors that are relevant for the lowlying nucleon and delta resonances with $L \leq 2$.

The matrix elements of Eq. (8.1) for vibrational excitations are obtained as above by projection from the corresponding intrinsic states. In Table II we present the results for the elementary form factors for the vibrational excitations $(0,0^0) \rightarrow (1,0^0)$, $(0,1^1)$ for nucleon and delta resonances with $L \leq 1$. The oblate top form factors in Tables I and II correspond to the transitions indicated schematically in Figure 4.

9 Summary and conclusions

In this contribution, we presented an algebraic treatment of the three-body problem. We used the method of bosonic quantization, which for the two relative coordinates of the three-body system gives rise to a $U(7)$ spectrum generating algebra. The model space is spanned by the symmetric irreps $[N]$ of $U(7)$.

In particular, we studied the case of three identical objects and showed how the corresponding permutation symmetry can be taken into account exactly. Rather than explicitly constructing states of good permutation symmetry [9], we obtain these states by diagonalizing a Hamiltonian that is invariant with respect to the point group symmetry. We developed a general procedure to determine the permutation symmetry of any given wave function. It was shown that in the large N limit (large model space) the algebraic Hamiltonian for the X_3 system corresponds to an oblate symmetric top.

For the special case of one- and two-body interactions, the eigenstates can be labeled by an additional quantum number M_F . This label plays a very interesting role. On the one hand, it has a direct connection to the permutation symmetry. On the other hand, in the large N limit it is directly related to the geometric labels K and l , and provides an extra label which is needed to classify the rovibrational states of the oblate top uniquely.

It was shown that $U(7)$ provides a unified treatment of both rotational and vibrational excitations of an oblate top. The ensuing algebraic treatment of the oblate top has found useful applications both

in molecular physics (X_3 molecules [6, 22]) and hadron physics (nonstrange qqq baryons [7, 20, 21]). As an example, we discussed an application to baryon resonances which are interpreted as vibrations and rotations of an oblate top. We derived closed expressions for a set of elementary form factors for nonstrange resonances. These elementary form factors appear in all calculations of electromagnetic, strong and weak couplings, and hence form the backbone of the model. In a separate contribution to these proceedings [23] we discuss in more detail the application to electromagnetic and strong couplings of baryon resonances.

Finally, we note that, although we discussed the specific case of three identical objects, the algebraic procedure is valid in general, both for three nonidentical objects and for the many-body system. Applications of these techniques to polyatomic molecules will be discussed in a separate contribution [24].

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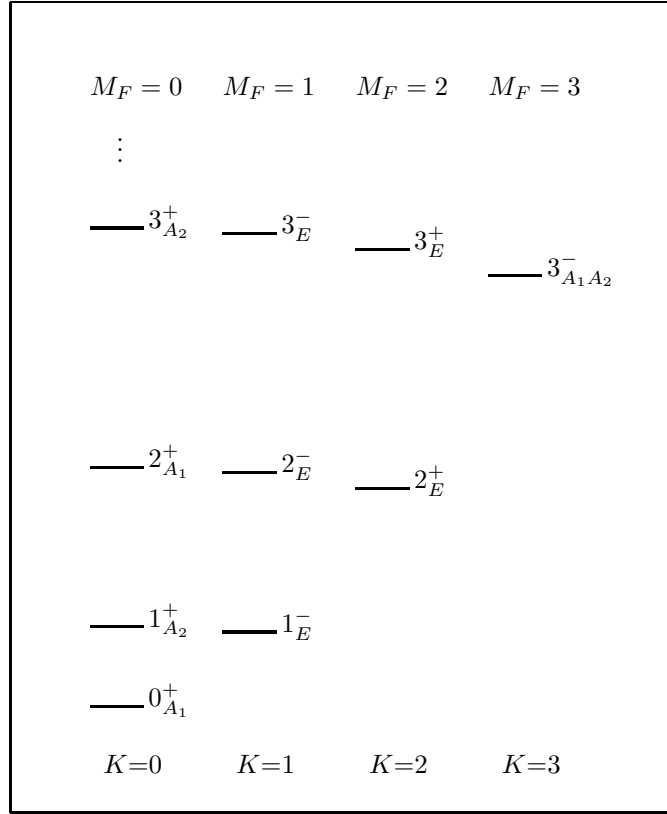


Figure 1: Schematic representation of rotational spectrum of a $(v_1, v_2^{l=0})$ vibrational band. The states are labeled by L_t^P . Here K denotes the absolute value of the projection of the angular momentum L on the symmetry axis.

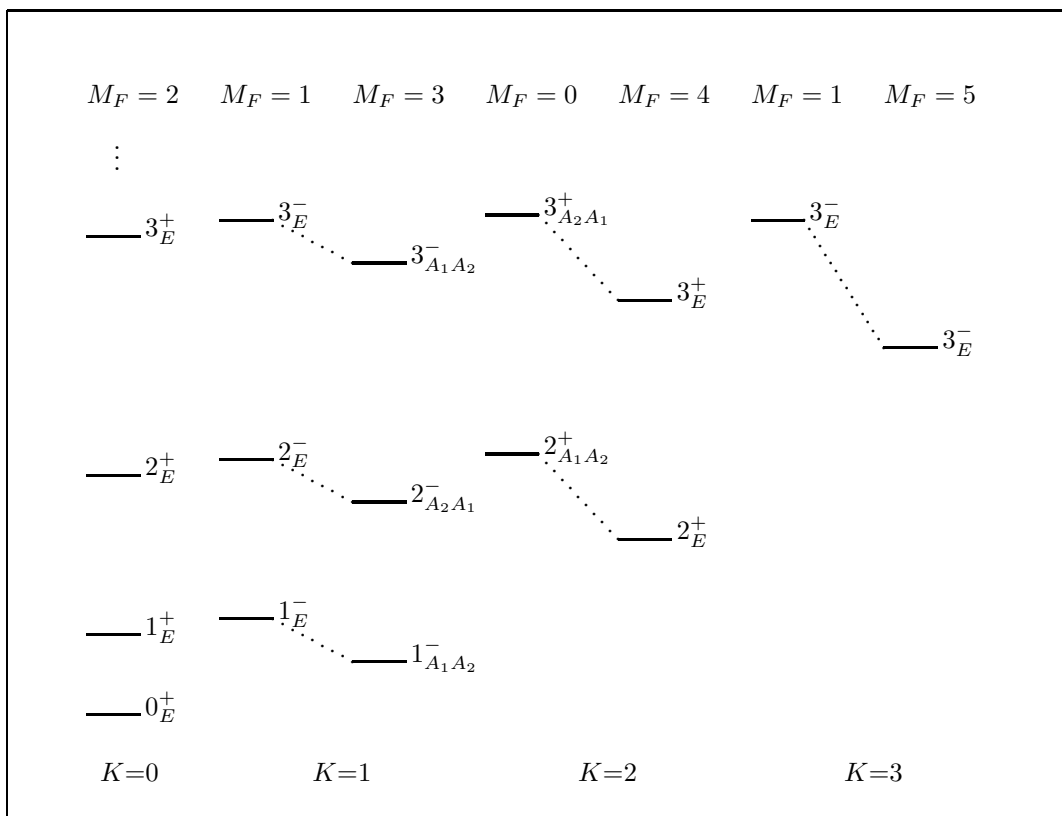


Figure 2: As Figure 1, but for a $(v_1, v_2^{l=1})$ vibrational band.

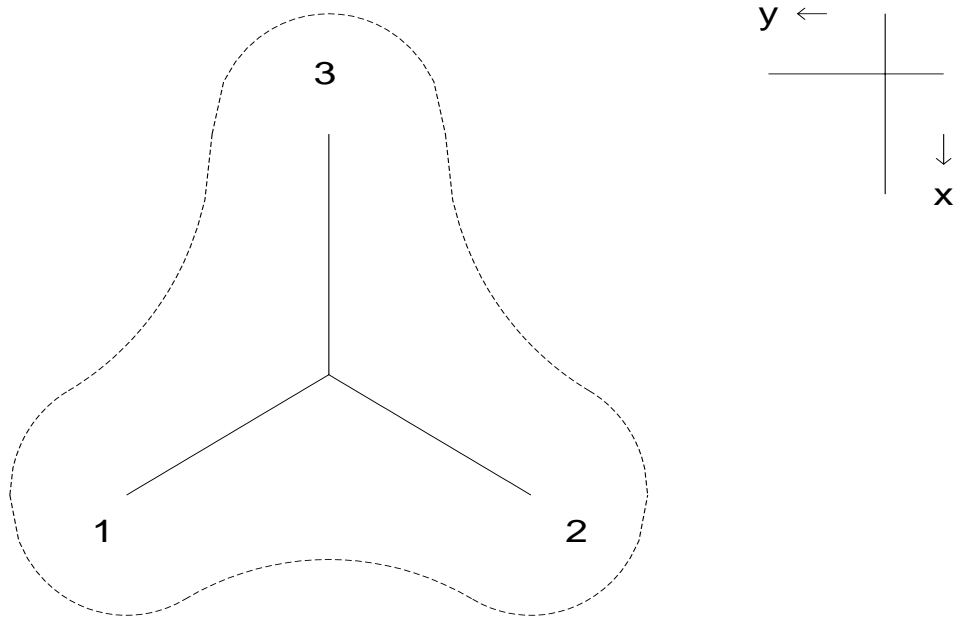


Figure 3: Geometry of collective model of baryons.

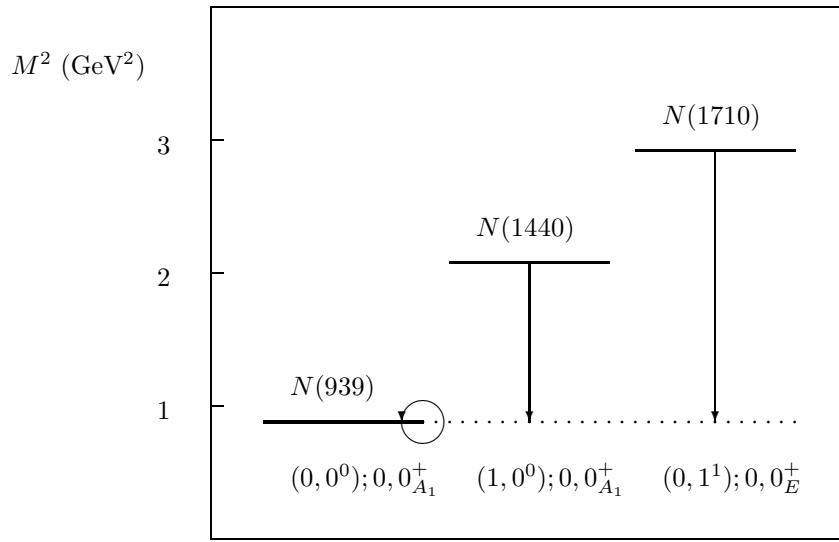


Figure 4: Schematic representation of the vibrational spectrum of nucleon resonances. The resonances are labeled by the usual spectroscopic notation [19] and their oblate top classification $(v_1, v_2^l); K, L_t^P$.

Table I: Analytic expressions of the $(v_1, v_2^l) = (0, 0^0) \rightarrow (0, 0^0)$ matrix elements of the transition operators of Eq. (8.1) in the oblate top model for $N \rightarrow \infty$. The final states are labeled by $[\dim\{SU_{sf}(6)\}, L^P]_{(v_1, v_2^l); K}$. The initial state is $[56, 0^+]_{(0, 0^0); 0}$.

Final state	$F_2(k)$	$G_{1;\rho,\pm}(k)/m_3 k_0 \beta$
$[56, 0^+]_{(0, 0^0); 0}$	$j_0(k\beta)$	0
$[20, 1^+]_{(0, 0^0); 0}$	0	$\frac{\sqrt{3}}{\sqrt{2}} j_1(k\beta)$
$[70, 1^-]_{(0, 0^0); 1}$	$-i\sqrt{3} j_1(k\beta)$	$\pm i \frac{1}{\sqrt{6}} [2j_0(k\beta) - j_2(k\beta)]$
$[56, 2^+]_{(0, 0^0); 0}$	$\frac{\sqrt{5}}{2} j_2(k\beta)$	0
$[70, 2^-]_{(0, 0^0); 1}$	0	$i \frac{\sqrt{5}}{\sqrt{2}} j_2(k\beta)$
$[70, 2^+]_{(0, 0^0); 2}$	$-\frac{\sqrt{15}}{2} j_2(k\beta)$	$\mp \frac{1}{\sqrt{10}} [3j_1(k\beta) - 2j_3(k\beta)]$

Table II: As Table II, but for the $(v_1, v_2^l) = (0, 0^0) \rightarrow (1, 0^0)$ and $(0, 1^1)$ transitions. $\chi_1 = (1 - R^2)/2R\sqrt{N}$ and $\chi_2 = -\sqrt{1 + R^2}/R\sqrt{2N}$.

Final state	$F_2(k)$	$G_{1,\rho,\pm}(k)/m_3 k_0 \beta$
$[56, 0^+]_{(1,0^0);0}$	$-\chi_1 k \beta j_1(k\beta)$	0
$[20, 1^+]_{(1,0^0);0}$	0	$\chi_1 \frac{1}{\sqrt{6}} k \beta [2j_0(k\beta) - j_2(k\beta)]$
$[70, 1^-]_{(1,0^0);1}$	$-i \chi_1 \frac{1}{\sqrt{3}} k \beta [j_0(k\beta) - 2j_2(k\beta)]$	$\pm i \chi_1 \frac{1}{\sqrt{6}} [2j_0(k\beta) + 2j_2(k\beta) - 3k\beta j_1(k\beta)]$
$[70, 0^+]_{(0,1^1);0}$	$-\chi_2 \frac{1}{\sqrt{2}} k \beta j_1(k\beta)$	0
$[70, 1^+]_{(0,1^1);0}$	0	$-\chi_2 \frac{\sqrt{3}}{2} k \beta j_2(k\beta)$
$[56, 1^-]_{(0,1^1);1}$	$-i \chi_2 \frac{\sqrt{3}}{2} k \beta j_2(k\beta)$	0
$[20, 1^-]_{(0,1^1);1}$	0	$\pm i \chi_2 \frac{\sqrt{3}}{2\sqrt{2}} [3j_2(k\beta) - k\beta j_1(k\beta)]$
	$F_4(k)$	$G_{3,\rho,\pm}(k)/m_3 k_0 \beta$
$[70, 1^-]_{(0,1^1);1}$	$-i \chi_2 \frac{1}{2\sqrt{3}} k \beta [2j_0(k\beta) - j_2(k\beta)]$	$\pm i \chi_2 \frac{1}{2\sqrt{6}} [j_2(k\beta) + 4j_0(k\beta) + 3k\beta j_1(k\beta)]$